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LYAPUNOV FUNCTIONS OF THE MECHANICAL ENERGY TYPE*

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When examining the properties of the stability and asymptotic behaviour of a system a Lyapunov function is often used as the total mechanical energy of the system /1—7/. By analogy with the division of the energy into kinetic and potential energy, it is proposed below to construct a Lyapunov function in the form of the sum of two subsidiary scalar functions, such that its derivative on account of the system is estimated using some kind of function of these subsidiary functions. Generalizing the results /8/, we examine the case when the derivative of the Lyapunov function can also take positive values, and the equation of comparison the emerges from the estimate of the Lyapunov function does not permit a separation of variables. V.V. Rumyantsev's theorem /3/ on the asymptotic stability with respect to the velocities of the equilibrium position of a dissipative mechanical system is generalized on the basis of the results obtained.

1. Consider the set of differential equations

$$x^{\cdot} = X(t, x) \quad (t \in R_+ = [0, \infty),\ x \in R^k) \tag{1.1}$$

where the function X is defined and continuous in the set $R_+ \times G$, where G is an open set.

Besides the standard notation and concepts /9/ we will use the following. The continuous function $\varphi: R_+ \to R_+$ is called positive on average /4/, if for any infinite system S of non-intersecting segments of the semi-axis R_+ of identical length we have the relation

$$\int_S \varphi(t)\, dt = \infty$$

We shall further introduce the notation $[a]_+ = \max\{0, a\}$ and $[a]_- = \max\{0, -a\}$ - the positive and negative parts of the real number a.

Theorem 1.1. We will assume that the continuously differentiable functions $V_1, V_2: R_+ \times G \to R$ and the continuous functions $\omega, r: R_+^2 \to R_+$, which satisfy the following conditions in the set $R_+ \times G$, exist:

$$1)\ V_1(t, x) \geqslant 0,\ V(t, x) = V_1(t, x) + V_2(t, x) \geqslant 0$$
$$2)\ V^{\cdot}(t, x) \leqslant -\omega(t, V_1(t, x)) + r(t, V(t, x))$$

where the functions $\omega(t, u), r(t, u)$ do not decrease with respect to u; for fixed values u_0 the function $\omega(t, u_0)$ is positive on the average and the solutions of the equation $u^{\cdot} = r(t, u)$ are bounded in R_+;

3) for any constant $\alpha, \alpha_1 > 0$ and for the continuous function $\xi: R_+ \to R^k$ from the inequality $V(t, \xi(t)) \leqslant \alpha\ (t \in R_+)$ it follows that the function

$$\int_{H(t)} [V_2^{\cdot}(s, \xi(s))]_- ds, \quad H(t) = \{s: 0 \leqslant s < t,\ \ V_1(s, \xi(s)) \geqslant \alpha_1\}$$

is uniformly continuous in R_+. Then for every solution $x(t)$ of Eq.(1.1), determined for all $t \geqslant t_0$

$$V_1(t, x(t)) \to 0,\ V_2(t, x(t)) \to \text{const} \quad (t \to \infty)$$

Proof. Suppose $x(t)$ is some solution of Eq.(1.1), determined in $[t_0, \infty)$, and suppose $u(t)$ is the upper solution of the problem $u^{\cdot} = r(t, u),\ u(t_0) = V(t_0, x(t_0))$.

The function $u(t)$ does not decrease and is bounded, and it therefore has the finite limit u_∞ as $t \to \infty$. We shall introduce the notation

$$v_1(t) = V_1(t, x(t)),\ v_2(t) = V_2(t, x(t))$$
$$v(t) = v_1(t) + v_2(t),\ w(t) = v(t) + u_\infty - u(t)$$

By virtue of condition 2) the function $v(t)$ satisfies the inequality $v^{\cdot} \leqslant r(t, v)$ in the

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interval (t_0, ∞); therefore using the fundamental comparison theorem /10/

$$w^{\cdot}(t) \leqslant -\omega(t, v_1(t)) \leqslant 0 \quad (t \geqslant t_0) \tag{1.2}$$

Thus, the function $w(t)$ is non-negative and does not increase, it therefore has it a finite limit as $t \to \infty$.

It remained to prove that $v_1(t) \to 0$ as $t \to \infty$. We shall assume the opposite, i.e. suppose

$$\limsup_{t\to\infty} v_1(t) > 0 \tag{1.3}$$

Integrating Eq.(1.2) in $[t_0, \infty)$, we obtain the inequality

$$\int_{t_0}^{\infty} \omega(t, v_1(t))\, dt \leqslant w(t_0) - \lim_{t\to\infty} w(t) < \infty \tag{1.4}$$

The function $\omega(t, u)$ is non-decreasing with respect to u and is positive on the average for fixed values of u; it therefore follows from (1.4) that

$$\liminf_{t\to\infty} v_1(t) = 0$$

Together with inequality (1.3) this entails the existence of a number $\gamma > 0$, such that for any $T \geqslant t_0$ the numbers $A = A(T)$, $B = B(T)$ $(T < A < B)$ which satisfy the relations $v_1(A) = 2\gamma/3$, $v_1(B) = \gamma/3$, $\gamma/3 \leqslant v_1(t) \leqslant 2\gamma/3$ for $t \in [A(T), B(T)]$ will be obtained.

By virtue of inequality (1.2) we have

$$w(B) - w(A) \leqslant -\int_A^B \omega(t, \gamma/3)\, dt$$

Since the function $\omega(t, \gamma/3)$ is positive on the average, then

$$\lim_{T\to\infty} (B(T) - A(T)) = 0 \tag{1.5}$$

On the other hand, the sum $v_1(t) + v_2(t)$ has a finite limit as $t \to \infty$, consequently $T_0 > 0$ exists, such that from $T > T_0$ it follows that $v_2(B(T)) - v_2(A(T)) > \gamma/6$. However then

$$\frac{\gamma}{6} \leqslant \int_{A(T)}^{B(T)} [v_2^{\cdot}(t)]_+\, dt \tag{1.6}$$

and this inequality (we take into account Eq.(1.5)) is obtained in contradiction to condition 3). Indeed, by virtue of condition 3) for every $\varepsilon > 0$ an $\delta(\varepsilon) > 0$ exists, such that from $0 < B(T) - A(T) < \delta$ there follows the inequality

$$0 \leqslant \int_A^B [v_2^{\cdot}(t)]_+\, dt = \int_A^B [V_2^{\cdot}(t, x(t))]_+\, dt < \varepsilon \tag{1.7}$$

which contradicts (1.5), (1.6). The theorem is proved.

Suppose $x = (y, z)$ is the splitting, for which $y \in R^m$, $z \in R^n$ $(0 < m \leqslant k, n = k - m)$, and we will assume that $0 \in G$. Using the above theorem to obtain the conditions of the asymptotic y-stability of the zero solution of Eq.(1.1), we obtain

Corollary 1.1. We will assume that all the conditions of Theorem 1.1 hold, whilst $V_2(t, x) \geqslant 0$, the function $V_1(t, y, z)$ is y-positive and the zero solution of the equation $u^{\cdot} = r(t, u)$ is stable. Then the zero solution of Eq.(1.1) is asymptotically y-stable.

Proof. Since $V^{\cdot}(t, x) \leqslant r(t, V(t, x))$ and the Lyapunov function $V(t, x)$ is y-positive definite, the zero solution of Eq.(1.1) is y-stable /9/. Hence using Theorem 1.1 we obtain the required statement.

Note 1.1. For simplicity Theorem 1.1 does not incorporate the strictest version of condition 3). It could be additionally assumed that the function $\xi(t)$ also has the property

$$\int_0^{\infty} \omega(t, V_1(t, \xi(t)))\, dt < \infty \tag{1.8}$$

(see (1.4)).

Theorem 1.1 also remains valid after replacing condition 3) by the following condition:

3') Suppose for any constants $\varepsilon, \alpha, \alpha_1 > 0$ and the continuous function $\xi\colon R_+ \to R^k$, which satisfy the relations

$$\sup_{t\in R_+} V(t,\xi(t))\leqslant\alpha,\quad \int_0^\infty \omega(t,V_1(t,\xi(t)))\,dt<\infty$$

$\delta=\delta(\varepsilon,\alpha,\alpha_1,\xi)>0$ exists, such that from the condition

$$A\leqslant B<A+\delta,\ \min_{t\in[A,B]} V_1(t,\ \xi(t))\geqslant\alpha_1$$

it follows that

$$\int_A^B [V_2^{\cdot}(t,\xi(t))]_+\,dt<\varepsilon$$

In fact, condition 3) was only used to derive estimate (1.7), and for this it is sufficient to require that 3') holds.

It is shown below that this condition of the theorem has practical value.

Note 1.2. An analysis of the proof of Theorem 1.1 also shows that in condition 3) it is sufficient to require the uniform continuity of the function

$$\int_{H(t)} [V_2^{\cdot}(s,\xi(s))]_- ds$$

in R_+, or of the function $V_2(t,\xi(t))$ in the set $H(\infty)$.

2. Consider a holonomic scleronomous mechanical system acted upon by potential, dissipative and gyroscopic forces

$$\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot}}-\frac{\partial T}{\partial q}=-\frac{\partial\pi}{\partial q}+Q+e(t),\quad q\in\Omega \tag{2.1}$$

$$T=T(q,q^{\cdot})=\frac{1}{2}(q^{\cdot})^T A(q)q^{\cdot}$$

where $\Omega\subset R^r$ is some open connected set, T is the kinetic energy (here and henceforth x^T signifies the transposed column vector $x\in R^r$), $\pi=\pi(t,q)$ is the potential energy ($\pi(t,0)\equiv0$), $Q=Q(t,q,q^{\cdot})$ the resultant of the gyroscopic and dissipative forces, i.e. $Q^T(t,q,q^{\cdot})q^{\cdot}\leqslant0$ for all $t\in R_+$, $q\in\Omega$, $q^{\cdot}\in R^r$.

We will assume that all the function in (2.1) are fairly smooth and all motions with fairly small initial values $\|q_0\|$, $\|q_0^{\cdot}\|$ are determined for all values of $t\geqslant t_0$. We shall examine the asymptotic behaviour of the generalized velocities $q^{\cdot}$.

For the stationary case $\pi=\pi(q)$, $Q=Q(q,q^{\cdot})$, $e(t)\equiv0$ it is proved /3/ that if the dissipation is complete and there is no potential force ($\pi(q)\equiv0$), then the equilibrium position $q=q^{\cdot}=0$ is asymptotically stable with respect to the velocities. Experience suggests that this is also true when $\pi(q)\geqslant0$ if the motion is bounded. This will be proved below using Theorem 1.1. In addition, we will generalize Rumyantsev's theorem /3/ to the non-stationary case. In all its previous generalizations /4, 5, 8/ the authors required that the quantity $\|Q(t,q,q^{\cdot})\|$, like the function t, is bounded in one sense or another. This condition unnatural since as the dissipation increases the velocity decreases. Using the results of Sect.1, we also obtain a theorem on the asyptotic $q^{\cdot}$-stability in the non-stationary case, which does not contain any requirement of the boundedness of $\|Q(t,q,q^{\cdot})\|$.

We shall introduce the notation

$$E_h(t)=\{q\in\Omega:\ \pi(t,q)<h\},\ E_h^*=\{(t,q)\in R^{r+1}:\ t\in R_+,\ q\in E_h(t)\}$$

Theorem 2.1. We will assume that for the arbitrary fixed value $h>0$ the following conditions hold for all $(t,q,q^{\cdot})\in E_h^*\times R^r$:

a) $T(q,q^{\cdot})\geqslant c\|q^{\cdot}\|^2$ $(0<c=c(h)=\text{const})$

b) $Q^T(t,q,q^{\cdot})q^{\cdot}\leqslant-\varphi(t)\,b(T(q,q^{\cdot}))$

where $\varphi, b: R_+\to R_+$ are continuous functions, φ is a positive function on the average, b is a strictly increasing function and $b(0)=0$;

c) $\pi(t,q)\geqslant0$

d) $[\partial\pi(t,q)/\partial t]_+\leqslant r(t,\pi(t,q))$

where the function $r(t,u)$ is continuous and increases with respect to u;

e) the solutions of the equation

$$u^{\cdot}=|e(t)|\,c^{-1/2}u^{1/2}+r(t,u) \tag{2.2}$$

with fairly small initial values are bounded;

f) the function $\|\partial\pi(t,q)/\partial q\|$ is bounded in the set E_h^*.

Then for every motion $q(t)$ with fairly small initial values we have $q^{\cdot}(t)\to0$, $\pi(t,q(t))\to\text{const}$ $(t\to\infty)$.

Proof. Suppose $V_1 = T,\ V_2 = \pi,\ V = V_1 + V_2$. Then

$$V^{\cdot}(t, q, q^{\cdot}) = \partial\pi/\partial t + Q^T q^{\cdot} + e(t)\, q^{\cdot} \leqslant -\varphi(t)\, b\,(V_1(q, q^{\cdot})) + |e(t)|\, c^{-1/2} V^{1/2} + r(t, V)$$

Suppose now $u(t) = u(t;\ t_0, u_0)$ is a solution of Eq.(2.2), such that $u_0 > 0,\ u(t) \leqslant h = \text{const}$ when $t \geqslant t_0$, and suppose $q = q(t)$ is the motion of system (2.1) with the initial values $q_0, q_0^{\cdot}$, which satisfy the inequality

$$V(t_0, q_0, q_0^{\cdot}) = T(q_0, q_0^{\cdot}) + \pi(t_0, q_0) < u_0$$

Using the comparison Theorem /10/

$$\pi(t, q(t)) \leqslant V(t, q(t), q^{\cdot}(t)) \leqslant u(t) \leqslant h \quad (t \geqslant t_0)$$

We shall now use Theorem 1.1. Conditions 1) and 2) obviously hold and it remains to verify that condition 3) holds for $\xi(t) = (q(t), q^{\cdot}(t))^T$. By virtue of the conditions of the theorem the following estimate holds:

$$[\pi^{\cdot}(t, q)]_+ \leqslant [\partial\pi/\partial t]_+ + \|\partial\pi/\partial q\|\, \|q^{\cdot}\| \leqslant r(t, u(t)) + \text{const}\,(h/c)^{1/2} \tag{2.3}$$

and consequently

$$\int_A^B [\pi^{\cdot}(t, q(t))]_+\, dt \leqslant u(B) - u(A) + \text{const}\,(B - A) \to 0, \quad (B - A) \to 0$$

Hence bearing in mind the fact that $u(t) \to u_\infty < \infty$, we directly obtain condition 3). The theorem is proved.

Consider now the case of the potential energy $\pi = \pi(q)$, which does not depend on the time t. We will examine only the bounded motion. (Since $\pi(q(t))$ has a finite limit, then for the boundedness of all the motion it is sufficient that the conditions $\pi(q) \to \infty$ $(\|q\| \to \infty)$ hold. The set E_h^* can now be replaced by the set $R_+ \times K$, where $K \subset R^r$ is a compactum, and since conditions a),d)-f) obviously hold, we obtain

Corollary 2.1. We will assume that in system (2.1) the potential energy does not depend on the time and is non-negative in the neighbhourhood of the equilibrium position $q = 0$. We shall further assume that the dissipation is complete on the average, i.e. for every compactum $K \subset R^r$ the continuous functions $\varphi, a\colon R_+ \to R_+$ exist, such that φ is positive on the average, the function a strictly increases, $a(0) = 0$ and $Q^T(t, q, q^{\cdot})\, q^{\cdot} \leqslant -\varphi(t)\, a(\|q^{\cdot}\|)$ in the set $R_+ \times K \times R^r$.

Then for every bounded motion $q(t)$ of system (2.1) the relations $q^{\cdot}(t) \to 0,\ \pi(q(t)) \to \text{const}$ $(t \to \infty)$ hold.

Corollary 2.2. We will assume that $\pi(q) \geqslant 0$ and every motion of the system

$$\frac{d}{dt}\frac{\partial T(q, q^{\cdot})}{\partial q^{\cdot}} - \frac{\partial T(q, q^{\cdot})}{\partial q} = -\frac{\partial \pi(q)}{\partial q} + Q(t, q, q^{\cdot}) \tag{2.4}$$

with fairly small initial values is bounded in R_+. We shall further assume that the dissipation is complete on the average (see Corollary 2.1).

Then the equilibrium position $q = q^{\cdot} = 0$ is asymptotically $q^{\cdot}$-stable.

We shall now formulate one corollary of Theorem 2.1 for Eq.(2.4), which does not require the boundedness of the motion.

We shall introduce the notation $E_h = \{q \in \Omega\colon \pi(q) \leqslant h\}$.

Corollary 2.3. We shall assume that for the arbitrary fixed value $h > 0$ the following conditions hold:

a) positive constants $c_1 = c_1(h),\ c_2 = c_2(h)$ exist, such that $c_1\|q^{\cdot}\|^2 \leqslant T(q, q^{\cdot}) \leqslant c_2\|q^{\cdot}\|^2$ $(q \in E_h,\ q^{\cdot} \in R^r)$;

b) the dissipation is complete, i.e. the continuous, strictly increasing function $a(r)$ exists, such that $a(0) = 0$ and $Q^T(t, q, q^{\cdot})\, q^{\cdot} \leqslant -a(\|q^{\cdot}\|)$ $((t, q, q^{\cdot}) \in R_+ \times E_h \times R^r)$;

c) $\pi(q) \geqslant 0$;

d) the function $\operatorname{grad} \pi(q)$ is bounded in the set E_h.

Then the equilibrium position $q = q^{\cdot} = 0$ of system (2.4) is asymptotically $q^{\cdot}$-stable.

Using Note 1.1, we can extend Theorem 2.1 to the case when the function $\partial\pi(t, q)/\partial q$ is unbounded in the set E_h^*. In fact conditionf f) in Theorem 2.1 can be replaced by the following:

f') for every $h > 0$ the function

$$\int_t^{t+1} \frac{1}{\varphi(s)} \max\left\{\left\|\frac{\partial \pi(q, s)}{\partial q}\right\|^2 : q \in E_h(s)\right\} ds$$

is bounded in R_+.

In fact, the proof of Theorem 2.1 ought to be modified only at the very end, but we need to show that (instead of condition 3)) condition 3') holds for $\xi(t) = (q(t), q^{\cdot}(t))^T$.

It is well-known that $\sup\limits_{t \geqslant t_0} V(t, q(t), q^{\cdot}(t)) \leqslant h$

$$\int_{t_0}^{\infty} \varphi(t)\, b(V_1(q(t), q^{\cdot}(t)))\, dt < \infty \tag{2.5}$$

We will assume that $V_1(q(t), q^{\cdot}(t)) \geqslant \alpha_1 > 0$ in the segment $[A, B]$ $(A < B < A + 1)$. From the estimate

$$[\pi^{\cdot}(t, q(t))]_+ \leqslant [\partial\pi(t, q(t))/\partial t]_+ + |(\partial\pi(t, q(t))/\partial q)^T q^{\cdot}| \leqslant$$
$$r(t, u(t)) + \max\{\|\partial\pi(t, q)/\partial q\| : q \in E_h(t)\} \times$$
$$(h/c)^{1/2}\varphi^{-1/2}(t)\,[\varphi(t)\, b(T(q, (t), q^{\cdot}(t)))]^{1/2}$$

using the Hölder inequality and condition f'), we obtain the inequality

$$\int_A^B [V_2^{\cdot}(t, q(t), q^{\cdot}(t))]_+ dt \leqslant u(B) - u(A) + \mathrm{const}\int_A^B \varphi(t)\, b(V_1(q(t), q^{\cdot}(t)))\, dt$$

By virtue of (2.5) and the relation $u(t) \uparrow u_\infty < \infty$ the right-hand side of the last estimate is arbitrarily small if the quantity $B - A$ is fairly small, which gives condition 3').

REFERENCES

1. LYAPUNOV A.M., The general problem of the stability of motion. Moscow-Leningrad: Gostekhizdat, 1950.
2. RUMYANTSEV V.V., The stability of motion in relation to part of the variables. Vestn. MGU, Matem. mekhan. astron., fiz., khim., 4, 1957.
3. RUMYANTSEV V.V., The asymptotic stability and instability of motion with respect to part of the variables. PMM, 35, 1, 1971.
4. MATROSOV V.M., The stability of motion. PMM, 26, 5, 1962.
5. OZIRANER A.S., Asymptotic stability and instability with respect to part of the variables. PMM, 37, 4, 1973.
6. TEREKI I. and HATVANI L., Partial stability and the convergence of motion. PMM, 45, 3, 1981.
7. TEREKI I. and HATVANI L., Asymptotic discontinuation when there is viscous friction. PMM, 46, 1, 1982.
8. HATVANI L., A generalization of the Barbashin - Krasovskij theorems to the partial stability in non-autonomous systems. Colloquia Math. Soc. J. Bolyai, 30. Qualitative Theory of Differential Equations, Szeged, Hungary, 1979.
9. OZIRANER A.S. and RUMYANTSEV V.V., The method of Lyapunov functions in the problem of the stability of motion with respect to part of the variables. PMM, 36, 2, 1972.
10. MATROSOV V.M., The principle of comparison with the Lyapunov vector-function. I-IV. Differents. uravneniya, 4, 8, 1968, 4, 10, 1968, 5, 7, 1969, 5, 12, 1969.

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